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On "Best" Interpolation*

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Recent interest in the problem of minimizing $\|f^{(k)}\|_\infty$ under the constraint that $f(t_i) = f_0(t_i)$, $i = 1,..., n + k$, for some given f_0 and given $(t_i)_1^{n+k}$ seems to make it worthwhile to explain how Favard solved this problem in the thirties, particularly since Favard's paper on the subject is rather sketchy in places.

The explanation is given in terms of a dual problem, using a technique initiated by M. Krein. In addition, the analogous problem of minimizing $\|f^{(k)}\|_p$ for $1 \leqslant p < \infty$ under similar constraints is discussed.

1. INTRODUCTION

In [2], Favard considers the problem of minimizing $f^{(k)}$ over

$$F := \{f \in \mathbb{L}_\infty^{(k)} \mid f(t_i) = f_0(t_i),\ i = 1,..., n + k\}$$

for given f_0 and given $0 \leqslant t_1 < t_2 < \cdots < t_{n+k} \leqslant 1$. Favard solves this problem in the following sense. He constructs a function $\hat{f} \in F$ at which $\inf_{f \in F} \|f^{(k)}\|_\infty$ is taken on and which, in addition, has the property that

$$f \in F \text{ and } |f^{(k)}(t)| \leqslant |\hat{f}^{(k)}(t)| \text{ for all } t \in [0, 1] \text{ implies } f = \hat{f}, \tag{1}$$

i.e., $\hat{f}$ is, in this sense, an element of F with absolutely smallest kth derivative. Favard also states that $\hat{f}$ is unique, but it is not clear exactly in what sense $\hat{f}$ is supposed to be unique. Certainly, (1) can hold for many different elements $\hat{f}$ of F, all of which also minimize $\|f^{(k)}\|_\infty$. But it is true that Favard's $\hat{f}$ is the unique solution of a certain intrinsic sequence of minimization problems, as a consequence of which $\hat{f}$ also satisfies (1).

It is the purpose of this paper to give a simple account of Favard's arguments using duality, and to discuss the analogous problem in $\mathbb{L}_p^{(k)}$ for $1 \leqslant p < \infty$ as well.

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2. Description and Reduction of the Problem

Let $\mathbf{t} := (t_i)$ be nondecreasing. For a sufficiently smooth f, denote by

$$f|_{\mathbf{t}} := (f_i)$$

the corresponding sequence given by the rule

$$f_i := f^{(j)}(t_i) \qquad \text{with} \qquad j := j(i) := \max\{m \mid t_{i-m} = t_i\}.$$

Assuming that $\operatorname{ran} \mathbf{t} \subseteq [a, b]$ and that $t_i < t_{i+k}$, all i, $f|_{\mathbf{t}}$ is defined for every f in the Sobolev space

$$\mathbb{L}_p^{(k)}[a, b] := \{f \in C^{(k-1)}[a, b] \mid f^{(k-1)} \text{ abs. cont.}; f^{(k)} \in \mathbb{L}_p[a, b]\},$$

with $1 \leqslant p \leqslant \infty$.

Consider the problem of minimizing $\|f^{(k)}\|_p$ over

$$F := F(\mathbf{t}, \boldsymbol{\alpha}, k, p, [a, b]) := \{f \in \mathbb{L}_p^{(k)}[a, b] \mid f|_{\mathbf{t}} = \boldsymbol{\alpha}\}$$

for some given $\boldsymbol{\alpha}$, some bounded $[a, b]$, and some finite $\mathbf{t}$. Then F is certainly not empty; it is, e.g., known that F contains exactly one polynomial of degree $<n + k$. Hence,

$$F = \{f \in \mathbb{L}_p^{(k)}[a, b] \mid f|_{\mathbf{t}} = f_0|_{\mathbf{t}}\}$$

for some fixed function $f_0 \in F$. Further,

$$[t_i, \ldots, t_{i+k}]f = [t_i, \ldots, t_{i+k}]f_0, \qquad \text{for all} \quad f \in F,$$

where

$$[t_i, \ldots, t_{i+k}]f$$

denotes the kth divided difference of f at the points $t_i, \ldots, t_{i+k}$. Favard already observes (without using the term "spline," of course) that, for $f \in \mathbb{L}_1^{(k)}[a, b]$,

$$[t_i, \ldots, t_{i+k}]f = \int_a^b M_{i,k}(t) f^{(k)}(t)\, dt/k!$$

with

$$M_{i,k}(t)/k! := [t_i, \ldots, t_{i+k}]\, (\cdot - t)_+^{k-1}/(k-1)!$$

a (polynomial) B-spline of order k having the knots $t_i, \ldots, t_{i+k}$. Hence, F is contained in

$$\left\{f \in \mathbb{L}_p^{(k)} \mid \int_a^b M_{i,k}(t) f^{(k)}(t)\, dt = \int_a^b M_{i,k}(t) f_0^{(k)}(t)\, dt,\ i = 1, \ldots, n\right\}. \tag{2}$$

On the other hand, for every f in the set (2), there exists a (unique) polynomial p_f of degree $<k$ so that $f - p_f \in F$, viz the unique polynomial p_f of degree $<k$ for which

$$p_f\,|_{(t_i)_1^k} = (f - f_0)|_{(t_i)_1^k}\,.$$

Consequently, with

$$\begin{aligned} g_0 &:= f_0^{(k)}, \\ G &:= G(\mathbf{t}, \boldsymbol{\alpha}, k, p, [a, b]) \\ &:= \left\{ g \in \mathbb{L}_p[a, b] \mid \int M_{i,k} g = \int M_{i,k} g_0\,, \quad i = 1,..., n \right\}, \end{aligned}$$

it follows that

$$\inf_{f \in F} \| f^{(k)} \|_p = \inf_{g \in G} \| g \|_p \tag{3}$$

and that k-fold differentiation maps the solution set of the left-hand minimization problem one–one and onto the set of solutions of the right-hand minimization problem. For this reason, Favard considers in detail minimization of $\| g \|_\infty$ over

$$\left\{ g \in \mathbb{L}_\infty[a, b] \mid \int \varphi_i g = \int \varphi_i g_0\,, \quad i = 1,..., n \right\}$$

with $\varphi_1 ,..., \varphi_n$ some integrable functions, as described in the next section.

3. An Interpretation of Favard's Arguments

Let $1 < p \leqslant \infty$, $1/p + 1/q = 1$. With $\varphi_1 ,..., \varphi_n \in \mathbb{L}_q[a, b]$, and $g_0 \in \mathbb{L}_p[a, b]$ given, let

$$G := \left\{ g \in \mathbb{L}_p[a, b] \mid \int \varphi_i g = \int \varphi_i g_0\,, \quad i = 1,..., n \right\}.$$

Further, let λ be the linear functional defined on

$$S := \operatorname{span}(\varphi_1 ,..., \varphi_n) \subseteq \mathbb{L}_q[a, b]$$

by the rule

$$\lambda\varphi := \int_a^b \varphi(t)\, g_0(t)\, dt, \qquad \text{for all} \quad \varphi \in S.$$

Then $\dim S \leqslant n$. If, in fact, $\dim S = 0$, the extremum problem has the unique solution $g = 0$. Otherwise, identifying $\mathbb{L}_p[a, b]$ in the usual way with the dual of $\mathbb{L}_q[a, b]$, G is seen to coincide with the collection of all extensions of λ to a continuous linear functional on $\mathbb{L}_q$. Hence,

$$\inf_{g\in G} \| g \|_p = \inf\{\| \mu \| \mid \mu \in (\mathbb{L}_q[a, b])^*, \mu|_S = \lambda\} = \| \lambda \|,$$

by the Hahn–Banach theorem, settling existence of a solution as well.

As to uniqueness and characterization of a minimum g, let ψ be an ($\mathbb{L}_q$-) extremal for λ, i.e.,

$$\psi \in S, \qquad \| \psi \|_q = 1, \qquad \lambda\psi = \sup_{\varphi\in S} \lambda\varphi/\| \varphi \|_q . \tag{4}$$

Then, for every minimum g,

$$\| g \|_p = \| \lambda \| = \lambda\psi = \int \psi g \leqslant \| \psi \|_q \| g \|_p = \| g \|_p .$$

Hence,

$$\int \psi g = \| \psi \|_q \| g \|_p . \tag{5}$$

For $p < \infty$ ($p = 1$ having been excluded at the outset), this implies that

$$g(t) = \| \lambda \| \, | \psi(t)|^{q-1} \operatorname{signum} \psi(t), \tag{6}$$

which characterizes the minimum g completely since $\| \cdot \|_q$ is strictly convex, hence ψ is uniquely determined by (4). In particular, $g = \| \lambda \| \, \psi$ for $p = 2$.

For $p = \infty$, and this is the case actually considered by Favard, (5) only implies that

$$g(t) = \| \lambda \| \operatorname{signum} \psi(t) \qquad \text{for} \quad t \notin N_\psi := \{t \mid \psi(t) = 0\}.$$

Actually, the slightly stronger statement holds that

$$h \in G \text{ and } | h | \leqslant | g | \text{ off } N_\psi \text{ implies } h = g \; (= \| \lambda \| \operatorname{sign} \psi) \text{ off } N_\psi . \tag{7}$$

In particular, the minimum g is unique in case $| N_\psi | = 0$.

If N_ψ has positive measure, then $\| \cdot \|_\infty$ may have more than one minimum in G, any two differing only on N_ψ. In this case, Favard apparently attempts to single out a particular minimum g_1 by the requirement that $g_1|_{N_\psi}$ also be a minimum for $\| \cdot \|_\infty$ in

$$G_\psi := \left\{ g \in \mathbb{L}_\infty(N_\psi) \;\middle|\; \int_{N_\psi} g\varphi = \int_{N_\psi} g_0\varphi, \text{ all } \varphi \in S \right\},$$

with g_0 now chosen in particular to be a minimum for $\|\cdot\|_\infty$ in G. This choice for g_0 guarantees that the minimal value of $\|\cdot\|_\infty$ on G_ψ is no bigger than $\|\lambda\|$, since it equals $\|\lambda_2\|$, with λ_2 the linear functional given on

$$S_\psi := \{\varphi|_{N_\psi} \mid \varphi \in S\}$$

by the rule

$$\lambda_2\varphi := \int_{N_\psi} g_0\varphi,$$

and so

$$\|\lambda_2\| \leqslant \|g_0\|_{\infty,N_\psi} \leqslant \|g_0\|_\infty = \|\lambda\|.$$

Further, any minimum g_2 of $\|\cdot\|_\infty$ in G_ψ gives rise to a minimum g_1 in G by the construction

$$g_1(t) := \begin{cases} g_2(t), & \text{for } t \in N_\psi, \\ g_0(t), & \text{for } t \notin N_\psi, \end{cases}$$

since evidently then $\|g_1\|_\infty = \|g_0\|_\infty$, and, for all $\varphi \in S$,

$$\int g_1\varphi = \int_{N_\psi} g_2\varphi + \int_{\setminus N_\psi} g_0\varphi = \int g_0\varphi,$$

as $g_2 \in G_\psi$. Finally,

$$\dim S_\psi < \dim S$$

since the linear map $\varphi \mapsto \varphi|_{N_\psi}$ takes the nonzero element ψ of S to 0. This implies that, while $\|\cdot\|_\infty$ may again not have a unique minimum in G_ψ, the resulting introduction of further additional minimization problems in the manner described will terminate after $\leqslant n$ steps in a problem on some N for which $\dim\{\varphi|_N \mid \varphi \in S\} = 0$.

The entire procedure is described more formally as follows.

Favard's Procedure

Let a bounded measurable set $N_1 \subseteq \mathbb{R}$, $\varphi_1,..., \varphi_n \in \mathbb{L}_1(N_1)$, $g_0 \in \mathbb{L}_\infty(N_1)$ be given. Set $S := \operatorname{span}(\varphi_1,..., \varphi_n) \subseteq \mathbb{L}_1(N_1)$.

Step 1. Set $i := 1$, $\hat{g} := g_0$.

Step 2. Set $m_i := \inf\{\|g\|_\infty \mid g \in G_i\}$ with

$$G_i := \left\{g \in \mathbb{L}_\infty(N_i) \mid \int_{N_i} g\varphi = \int_{N_i} \hat{g}\varphi, \text{ all } \varphi \in S\right\}$$

and pick a point g_i in G_i at which the infimum is attained.

Step 3. If $S_i := \{\varphi|_{N_i} \mid \varphi \in S\}$ has dimension 0, set $N_{i+1} := \varnothing$. Otherwise, pick $\psi_i \in S$ so that $\psi_i|_{N_i}$ is an $\mathbb{L}_1$-extremal for $\lambda_i: S_i \to \mathbb{R}: \varphi \mapsto \int_{N_i} \hat{g}\varphi$, i.e.,

$$\psi_i \in S, \| \psi_i \|_{1,N_i} = 1, \int_{N_i} \hat{g}\psi_i = \| \lambda_i \| = \sup_{\varphi \in S} \int_{N_i} \hat{g}\varphi / \| \varphi \|_{1,N_i}$$

and set $N_{i+1} := \{t \in N_i \mid \psi_i(t) = 0\}$.

Step 4. Redefine $\hat{g}$ to equal g_i on N_i.

Step 5. If $| N_{i+1} | > 0$, increase i by 1 and go to Step 2. Otherwise, stop.

In the terms and notation already introduced, the earlier discussion implies the following

LEMMA 1. *Favard's procedure produces a function $\hat{g} \in G_1$ with $\| \hat{g} \|_\infty = \inf\{\| g \|_\infty \mid g \in G_1\}$, and decreasing sequences $N_1 \supseteq \cdots \supseteq N_m$ and $m_1 \geqslant \cdots \geqslant m_m \geqslant 0$ for some $m \leqslant n$ so that, for $i = 1,\ldots, m$, $| \hat{g} | = m_i$ on $N_i \backslash N_{i+1}$,*

$$g \in G_1 \text{ and } | g | \leqslant | \hat{g} | \text{ on } N_1\backslash N_{i+1} \text{ implies that } g = \hat{g} \text{ on } N_1\backslash N_{i+1}, \tag{8}$$

(with N_{m+1} some set of measure zero).

Favard's procedure involves a certain number of choices, each of which could, offhand, affect the final output. The following, independent characterization of $\hat{g}$ is therefore of interest.

LEMMA 2. *For given $g_0 \in \mathbb{L}_\infty[a, b]$, given finite-dimensional subspace $S \subseteq \mathbb{L}_1[a, b]$, and given (measurable) $N_1 \subseteq [a, b]$, let*

$$E(g_0, S, N_1)$$

consist of exactly those $\tilde{g} \in G := \{g \in \mathbb{L}_1(N_1) \mid \int \varphi g = \int \varphi g_0, \text{ all } \varphi \in S\}$ which satisfy the following.

(i) *$| \tilde{g} |$ has finite (essential) range, $\{m_1, \ldots, m_r\}$ say, with $m_1 > \cdots > m_r$.*

(ii) *With $N_i := \{t \in N_1 \mid | \tilde{g}(t)| \leqslant m_i\}$, $i = 1,\ldots, r$, $N_{r+1} := \varnothing$, and, for $i = 1,\ldots, r$, $\tilde{g}$ uniquely minimizes $| g |$ on $N_1\backslash N_{i+1}$, i.e., $g \in G$ and $| g | \leqslant | \tilde{g} |$ on $N_1\backslash N_{i+1}$ implies that $g = \tilde{g}$ on $N_1\backslash N_{i+1}$.*

Then $E(g_0, S, N_1)$ has exactly one element, viz. the function produced from g_0, S, and N_1 by Favard's procedure.

Proof. Lemma 1 already establishes that $\hat{g} \in E(g_0, S, N_1)$ for any $\hat{g}$ produced from given g_0, S, and N_1 by Favard's procedure. It therefore suffices to show that $E(g_0, S, N_1)$ cannot have more than one element. This we prove by induction on $\dim S_1$ with $S_1 := \{\varphi |_{N_1} \mid \varphi \in S\}$, it being trivially true when $\dim S_1 = 0$. Hence assume that $\dim S_1 > 0$, and that, in addition to some $\tilde{g}$ satisfying (i) and (ii) above, $E(g_0, S, N_1)$ also contains a certain $\tilde{g}'$, with $\text{ess.ran}|\tilde{g}'| = \{m_1', \ldots, m_{r'}'\}$, $m_1' > \cdots > m_{r'}'$, and let $N_i' := \{t \in N_1 \mid |\tilde{g}'(t)| \leqslant m_i'\}$, $i = 1, \ldots, r'$; $N_{r'+1}' := \varnothing$. Assume without loss of generality that $m_1' \leqslant m_1$. Then $|\tilde{g}'| \leqslant |\tilde{g}|$ on $N_1 \backslash N_2$, hence $\tilde{g}' = \tilde{g}$ on $N_1 \backslash N_2$, therefore $m_1' = m_1$ and $N_2' \subseteq N_2$; but then $|\tilde{g}| \leqslant |\tilde{g}'|$ on $N_1 \backslash N_2'$ and so $\tilde{g}' = \tilde{g}$ on $N_1 \backslash N_2'$ showing that $N_2 = N_2'$. It follows that $\int_{N_2} \tilde{g}\varphi = \int_{N_2} \tilde{g}'\varphi$, all $\varphi \in S$, hence both $\tilde{g}|_{N_2}$ and $\tilde{g}'|_{N_2}$ are in $E(\tilde{g}, S, N_2)$ and, as $\dim\{\varphi|_{N_2} \mid \varphi \in S\} < \dim S_1$, induction hypothesis therefore gives $\tilde{g} = \tilde{g}'$ also on N_2. ∎

It follows in particular that

$$g \in G_1 \text{ and } |g| \leqslant |\hat{g}| \text{ implies } g = \hat{g}. \tag{9}$$

Hence, *$\hat{g}$ is smallest in absolute value among the elements of G.* But, contrary to what the italicized statement of [2, p. 289] might indicate, $\hat{g}$ need not be the only element of G_1 satisfying (9). Take, e.g.,

$$G_1 := \left\{ g \in \mathbb{L}_\infty[0, 1] \mid \int_0^1 g(t)\,dt = 1 \right\}.$$

Then, for every $s \in [0, 1)$, the function $\hat{g}(t) := (t - s)_+^0/(1 - s)$ is in G_1 and satisfies (9). To enlarge upon the example somewhat: For each $s \in [0, \frac{1}{2}]$,

$$\hat{g}(t) := \begin{cases} 0, & 0 < t < s, \\ 1/(1-s), & s < t < 1, \\ 2, & 1 < t < 2, \end{cases}$$

is a minimum for $\|\cdot\|_\infty$ in

$$G_1 := \left\{ g \in \mathbb{L}_\infty[0, 2] \mid \int_0^1 g(t)\,dt = 1, \int_1^2 g(t)\,dt = 2 \right\}$$

and satisfies (9).

It should be noted that the idea of looking at constrained minimization dually, as a problem of finding norm-preserving extensions for a given linear functional, and then using representation theorems for such functionals is far from new. The earliest and basic reference is apparently Krein who in [9], a publication predating Favard's paper, analyzed in this fashion what he calls *the L-moment problem*: To determine, for given $\varphi_1, \ldots, \varphi_n \in \mathbb{L}_1(N)$ and $(c_i)_1^n \in \mathbb{R}^n$, the numbers L so that there exist $g \in \mathbb{L}_\infty(N)$ with

$$\|g\|_\infty \leqslant L \quad \text{and} \quad \int_N \varphi_i g = c_i, \quad i = 1, \ldots, n.$$

But, in contrast to Favard, Krein is content to consider only the case where $S = \operatorname{span}(\varphi_i)_1^n$ has the happy property that all of its nonzero elements vanish only on a set of measure zero.

4. Discussion of Case $p = 1$

The case $p = 1$ demands special discussion since $\mathbb{L}_p$ fails to be the dual of $\mathbb{L}_q$ in this case. This is reflected in the fact that $\|\cdot\|_1$ may not have a minimum in

$$G = \left\{ g \in \mathbb{L}_1[a, b] \mid \int \varphi_i g = \int \varphi_i g_0, \quad i = 1,..., n \right\}.$$

Of course, with

$$S = \operatorname{span}(\varphi_1, ..., \varphi_n) \subseteq \mathbb{L}_\infty[a, b]$$

and

$$\lambda : S \to \mathbb{R} : \varphi \mapsto \int \varphi g_0,$$

as before, it continues to be true (as was observed and utilized in a much more general setting by Holmes [6]) that

$$\inf\{\| g \|_1 \mid g \in G\} = \inf\{\| \mu \| \mid \mu \in (\mathbb{L}_1[a, b])^*, \mu|_S = \lambda\} = \| \lambda \|,$$

by the Hahn–Banach theorem, which also gives the existence of solutions for the second problem. But, unfortunately, none of these solutions may be representable as integration against an $\mathbb{L}_1$-function. In this situation, Fisher and Jerome [5] recently proposed to restrict attention to the case $S \subseteq C[a, b]$, and, correspondingly, to look for those solutions of

$$\inf\{\| \mu \| \mid \mu \in C^*[a, b], \mu|_S = \lambda\}$$

which are extreme points of the convex set of all solutions. They show that all such extreme points must be of the form

$$\sum_{i=1}^{n} \beta_i \delta_{\xi_i},$$

with δ_t denoting point evaluation at t. This is not too surprising in view of the fact that $\{\pm\delta_t \mid t \in [a, b]\}$ comprises the extreme points of the unit ball of $C^*[a, b]$, while by a lemma by Singer [10; Chap. 2, Lemma 1.3] every linear functional λ on an n-dimensional subspace of a normed linear space X has a norm-preserving extension to all of X of the form

$$\sum_{i=1}^{n} \beta_i e_i ,$$

with $\beta_i \geqslant 0$ and e_i an extreme point of the unit ball of X^*, $i = 1,..., n$, and, of course, $\sum_i \beta_i = \| \lambda \|$.

It turns out that S is often merely in the space

$$C_\tau := C[\tau_0 , \tau_1] \times \cdots \times C[\tau_{m-1} , \tau_m]$$

of piecewise continuous functions on $[a, b]$ with breakpoints at $\tau_1 ,..., \tau_{m-1}$, for some $\boldsymbol{\tau} = (\tau_i)_0^m$ with $a = \tau_0 < \cdots < \tau_m = b$. The only additional point to be made then is that, strictly speaking, each $f \in C_\tau$ is defined on

$$[a, b]_\tau := [\tau_0^+, \tau_1^-] \cup \cdots \cup [\tau_{m-1}^+ , \tau_m^-],$$

i.e., f has the two values $f(\tau_i^-)$ and $f(\tau_i^+)$ "at τ_i." This implies that the extreme points of the unit ball of C_τ^* are all of the form $\pm\delta_t$ for some $t \in [a, b]_\tau$. Hence, Singer's lemma implies the

LEMMA. *Although, with $g_0 \in \mathbb{L}_1[a, b]$, and $\varphi_1 ,..., \varphi_n \in \mathbb{L}_\infty[a, b]$, $\| \cdot \|_1$ may fail to have a minimum in*

$$G := \left\{ g \in \mathbb{L}_1[a, b] \mid \int \varphi_i g = \int \varphi_i g_0 , \quad i = 1,..., n \right\}$$

even if $S := \operatorname{span}(\varphi_1 ,..., \varphi_n) \subseteq C[a, b]$, if $S \subseteq C_\tau$ for some $\boldsymbol{\tau} = (\tau_i)_0^m$ with $a = \tau_0 < \cdots < \tau_m = b$, then there exists $\boldsymbol{\beta} \in \mathbb{R}^n$ and $\xi_1 ,..., \xi_n \in [a, b]_\tau$ so that

$$\inf\{\| g \|_1 \mid g \in G\} = \sum_i \mid \beta_i \mid = \| \lambda \|$$

with

$$\lambda : S \to \mathbb{R} : \varphi \mapsto \int \varphi g_0 ,$$

while

$$\left(\sum_{i=1}^{n} \beta_i \delta_{\xi_i} \right) \Big|_S = \lambda.$$

Finally, note that the extremals for λ help once again to characterize at least partially the norm-preserving extensions of λ. Specifically, if

$$\sum_{i=1}^{r} \beta_i \delta_{\xi_i}$$

is some norm-preserving extension of λ to all of C_τ (with r an arbitrary integer), and ψ is any $\mathbb{L}_\infty$-extremal for λ, i.e.,

$$\psi \in S, \qquad \|\psi\|_\infty = 1, \qquad \lambda\psi = \|\lambda\|,$$

then

$$\|\lambda\| = \lambda\psi = \sum_{i=1}^{r} \beta_i \psi(\xi_i) \leqslant \sum_{i=1}^{r} |\beta_i| \, \|\psi\|_\infty = \|\lambda\|,$$

hence, for all i,

$$\beta_i \psi(\xi_i) = |\beta_i| \, \|\psi\|_\infty .$$

But this implies that every "active" ξ_i, i.e., every ξ_i with $\beta_i \neq 0$, must be an extreme point for every $\mathbb{L}_\infty$-extremal of λ. At times, this implies that $r \leqslant n$.

It should be noted that each such linear functional $\sum_i \beta_i \delta_{\xi_i}$, and in fact every continuous linear functional μ on C_τ, has a unique representation as a function $h \in NBV[a, b]_\tau$ in the sense that

$$\mu g = \int_a^b g \, dh, \qquad \text{for all} \quad g \in C_\tau .$$

Here, $NBV[a, b]_\tau$ is the Banach space of all functions h of bounded variation on $[a, b]$ with norm $\| dh \| = \mathrm{Var}(h)$, with each h normalized to be continuous from the right at every point except $\tau_0, \ldots, \tau_{m-1}$, and to satisfy $h(a) = 0$. The function h corresponding to $\sum_i \beta_i \delta_{\xi_i}$ is piecewise constant with jumps only at the points $\xi_1 \leqslant \cdots \leqslant \xi_r$, the jump at ξ_i having (signed) height β_i, unless $\xi_i = \tau_j^-$ and $\xi_{i+1} = \tau_j^+$, in which case h has a double jump at τ_j.

5. Existence and (Partial) Characterization of "best" Interpolants

We return to the original problem of minimizing $\| f^{(k)} \|_p$ over all f in

$$F_p = F(\mathbf{t}, f_0, k, p, [a, b]) = \{ f \in \mathbb{L}_p^{(k)}[a, b] \mid f|_{\mathbf{t}} = f_0|_{\mathbf{t}} \},$$

for some given $\mathbf{t} = (t_i)_1^{n+k}$ with $t_i < t_{i+k}$, and $t_i \in [a, b]$, all i, and some given f_0. We infer from Sections 2 and 3 that, for $1 < p < \infty$, $\| f^{(k)} \|_p$ has a unique minimum and that this minimum satisfies

$$f^{(k)} = |\psi|^{q-1} \operatorname{signum} \psi$$

for some $\psi \in S_{k,\mathbf{t}} := \operatorname{span}(M_{1,k}, \ldots, M_{n,k}) =:$ the linear space of *polynomial splines of order k with knot sequence* $\mathbf{t}$. Such matters as differentiability at the t_i's or at other points, or the number of possible zeros, or the fact that $f^{(k)} = 0$ off (t_1, t_{n+k}), etc., can therefore all be read off from known facts about polynomial splines. In the special case $p = 2$, one obtains, of course, $f^{(k)} = \psi$, hence f is a so-called natural spline of order $2k$ (at least in case $a < t_1$ and $t_{n+k} < b$).

Things become a bit more interesting and actually new in the case of $p = \infty$.

LEMMA. *With $N_1 = [a, b]$, $\varphi_i = M_{i,k}$, $i = 1,..., n$, Favard's procedure produces a step function $\hat{f}^{(k)} = \hat{g}$ with the following properties.*

(i) *$\hat{f}^{(k)}$ vanishes off (t_1, t_{n+k});*

(ii) *$|\hat{f}^{(k)}|$ has all its jumps at points of* $\mathbf{t}$;

(iii) *$\hat{f}^{(k)}$ has less than n jumps in (t_1, t_{n+k}).*

Proof. Let $\hat{g}$ be the step function produced by Favard's procedure together with the sequences $N_1 \supseteq N_2 \supseteq \cdots$, and $\psi_1, \psi_2,...,$ all in $S_1 = \mathbb{S}_{k,\mathbf{t}}$. $|\hat{g}|$ has jumps only at the essential boundary points of N_i for $i = 2, 3,...$. On the other hand,

$$N_i = \bigcap_{j<i} \{t \in N_1 \mid \psi_j(t) = 0\}.$$

Since the ψ_i's are piecewise polynomial with breakpoints only at the t_j's, each set $\{t \in N_1 \mid \psi_i(t) = 0\}$ consists, aside from $<n$ isolated points and the set $[a, b] \backslash [t_1, t_{n+k}]$, of a finite union of one or more disjoint intervals of the form $[t_L, t_R]$. This proves (i) and (ii).

As to (iii), we prove the slightly stronger statement,

With N_1 some interval of the form $[t_L, t_R]$ and $S_1 = \operatorname{span}\{M_{j,k} \mid j \in J\}$ for some subset J of $\{1,..., n\}$ such that

$$\bigcup_{j \in J} \operatorname{supp} M_{j,k} \supseteq N_1,$$

Favard's procedure produces a step function $\hat{g}$ having $<\dim S_1$ jumps,

by induction on $\dim S_1$, it being trivially true for $\dim S_1 \leqslant 1$. The well-known variation-diminishing property of splines discovered by Schoenberg (see, e.g., [7] for a proof) implies that ψ_1 has less than $\dim S_1$ strong sign changes on N_1; hence there is nothing more to prove unless $N_2 = \{t \in N_1 \mid \psi_1(t) = 0\}$ has positive measure. In that case, as already remarked upon above, N_2 is the union of one or more, say of r, mutually disjoint intervals

$$I_i := [t_{L_i}, t_{R_i}], \qquad i = 1,..., r.$$

Correspondingly, $N_1 \backslash N_2$ is the union of one or more, say of s, intervals J_i, $i = 1,\ldots, s$ (mutually disjoint), with $|r - s| \leqslant 1$. Together, $I_1,\ldots, I_r$, $J_1,\ldots, J_s$ give a partition of N_1. We claim that this partition induces a partition $\hat{I}_1,\ldots, \hat{I}_r, \hat{J}_1,\ldots, \hat{J}_s, \hat{K}$ of J by

$$\begin{aligned}\hat{I}_i &:= \{j \in J \mid |\operatorname{supp} M_{j,k} \cap I_i| > 0\},\\ \hat{J}_i &:= \{j \in J \mid \operatorname{supp} M_{j,k} \subset J_i\},\\ \hat{K} &:= \{j \in J \mid |\operatorname{supp} M_{j,k} \cap N_1| = 0\}.\end{aligned}$$

This is obviously a partition for J except perhaps for the fact that $\hat{I}_\mu \cap \hat{I}_\nu = \varnothing$ for all $\mu < \nu$. But, since all B-splines which do not vanish identically on a given interval are linearly independent there, it follows with $\psi_1 = \sum_{j\in J} \beta_j M_{j,k}$ that

$$\beta_j = 0 \text{ for all } j \in J \text{ for which } M_{j,k}|_{N_2} \neq 0.$$

If now $j \in \hat{I}_\mu \cap \hat{I}_\nu$, then every $M_{s,k}$ not identically zero on

$$(t_{R_\mu}, t_{L_\nu}) \subseteq \operatorname{supp} M_{j,k} = [t_j, t_{j+k}]$$

would be not identically zero on either I_μ or else on I_ν; hence $\beta_s = 0$ for each such s, which implies that ψ_1 vanishes identically between I_μ and I_ν, a contradiction.

It follows that

$$\dim S_1 = \sum_i |\hat{I}_i| + \sum_i |\hat{J}_i|$$

and that $S_2 = \{\varphi|_{N_2} \mid \varphi \in S_1\}$ breaks up into r mutually orthogonal subspaces

$$S_{2,i} := \{\varphi|_{I_i} \mid \varphi \in S_1\} = \operatorname{span}\{M_{j,k}|_{I_i} \mid j \in \hat{I}_i\}$$

hence $\hat{g}$ can be thought of as having been obtained on N_2 by applying Favard's procedure separately to each of these r component problems. Further, $\cup_{j\in I_i} \operatorname{supp} M_{j,k} \supseteq I_i$; hence $\dim S_{2,i} > 0$ and, by induction hypothesis,

$$\operatorname{jump}(I_i) \leqslant \dim S_{2,i} - 1 = |\hat{I}_i| - 1$$

with $\operatorname{jump}(I) :=$ number of jumps of $\hat{g}$ on I. Also, on J_i, ψ is in the span of $\{M_{j,k} \mid j \in \hat{J}_i\}$. Hence, by the variation-diminishing property of splines,

$$\operatorname{jump}(J_i) \leqslant |\hat{J}_i| - 1.$$

Hence, counting the $r + s - 1$ possible jumps in $\hat{g}$ between I_i's and J_i's,

$$\operatorname{jump}(N_1) \leqslant \sum_i (|\hat{I}_i| - 1) + \sum_i (|\hat{J}_i| - 1) + r + s - 1 = \dim S_1 - 1. \quad \blacksquare$$

We will call the unique $\hat{f}$ in

$$F_\infty = \{f \in \mathbb{L}_\infty^{(k)}[a, b] \mid f|_{\mathbf{t}} = \boldsymbol{\alpha}\},$$

for which $\hat{f}^{(k)}$ is the function produced by Favard's procedure (with $N_1 = [a, b]$, $\varphi_i = M_{i,k}$, $i = 1,..., n$) *Favard's solution* to the problem of minimizing $\|f^{(k)}\|_\infty$ over F_∞. We infer from the preceding lemma that Favard's solution is a polynomial spline of order $k + 1$ with kth derivative zero outside $[t_1, t_{n+k}]$, and less than n knots, all simple, inside (t_1, t_{n+k}).

Spline solutions which, in general, are different from Favard's solution have been identified by various authors. In his thesis [11], Smith showed the existence of a solution f with $f^{(k)}$ a step function having fewer than k jumps in each interval $[t_i, t_{i+1}]$, with $|f^{(k)}|$ having jumps only at the points of $\mathbf{t}$. Smith constructed such a solution as a limit point of the net $(f_p)_{p<\infty}$ of unique minimum points in F_p for $\|f^{(k)}\|_p$, as $p \to \infty$. It now seems likely that Favard's solution is such a limit point if not usually *the limit* of $(f_p)_p$ as $p \to \infty$.*

Karlin [8] was the first to see that at least one solution exists in the form of a *perfect* spline of order $k + 1$, i.e., a spline f of order $k + 1$ with $|f^{(k)}|$ constant, and, more significantly, that a perfect spline solution could be found having $<n$ knots (all simple). A simple proof can be found in [1].

Fisher and Jerome [3] showed the existence of an interval $[t_r, t_{r+k}]$ on which *all* solutions have the same kth derivative. This is not too surprising in view of the facts that all solutions must agree with Favard's solution on the set on which Favard's solution takes on its extreme values, and that this set contains the support of some nonzero element of $\mathbb{S}_{k,\mathbf{t}}$, hence the support of at least one B-spline $M_{r,k}$. It might be guessed from [3, Section 1] that there is a unique solution $f \in F_\infty$ (on $[t_1, t_{n+k}]$) with the further property that, on each subinterval $[t_i, t_{i+1}]$, $\|f^{(k)}\|_\infty$ is minimal. While it is true that there could be only one such solution, it is not true that such a solution always exists. If, e.g., $k = 2$, $t_i = i$, $i = 1,..., 5 = n + k$, and the conditions in terms of $f^{(2)}$ are

$$\int M_{1,2} f^{(2)} = 2, \qquad \int M_{2,2} f^{(2)} = 3, \qquad \int M_{3,2} f^{(2)} = 4,$$

then Favard's solution turns out to satisfy

$$\hat{f}^{(2)}(t) = \begin{cases} 2, & 1 < t < 3, \\ 4, & 3 < t < 5. \end{cases}$$

* *Added in proof:* This suggestion has been taken up by C. K. Chui, P. W. Smith, and J. D. Ward who proved in "Favard's solution is the limit of $W^{k,p}$-splines," to appear in *Trans. Amer. Math. Soc.*, that $(f_p)_{p<\infty}$ converges $\mathbb{L}_1^{(k)}$ to Favard's solution as $p \to \infty$.

On the other hand, the function $\tilde{f}$ with

$$\tilde{f}^{(2)}(t) = \begin{cases} 1, & 1 < t < 2, \\ 1 + 3(3 - t), & 2 < t < 3, \\ 4, & 3 < t < 5, \end{cases}$$

is also a minimum. If now f were a minimum of the kind described above, then it would follow that

$$| f^{(2)} | \leqslant 1 \text{ on } (1, 2), \qquad | f^{(2)} | \leqslant 2 \text{ on } (2, 3).$$

But then

$$\int M_{1,2} f^{(2)} \leqslant 3/2 < 2,$$

i.e., such f could not be in F_∞.

Finally, we consider the case $p = 1$, based on the discussion in Section 4. The elements of $S = \mathbb{S}_{k,\mathbf{t}}$ are all in C_τ, with τ consisting of a and b together with all points of $\mathbf{t}$ of multiplicity k.

Consider first the case that no point of $\mathbf{t}$ has multiplicity k. Then $\mathbb{S}_{k,\mathbf{t}} \subseteq C[a, b]$, and every $\psi \in \mathbb{S}_{k,\mathbf{t}}$ vanishes outside of (t_1, t_{n+k}). Hence, with $h \in NBV[a, b]_\tau$ such that $dh = \sum_1^n \beta_i \delta_{\xi_i}$ while

$$\int s\, dh = \int s\, df_0^{(k-1)}, \qquad \text{all} \quad s \in \mathbb{S}_{k,\mathbf{t}},$$

$$\| dh \| = \sum_i^n | \beta_i | = \inf\{\| df^{(k-1)} \| \; (= \| f^{(k)} \|_1) \mid f \in F_1\}$$

(the existence of which is assured by the discussion in Section 4), the function

$$f_h(t) := \int_a^b (t - u)_+^{k-1} \, (dh)(u)/(k - 1)!$$

is a polynomial spline of order k with (interior) knots $\xi_1, \ldots, \xi_n$, all simple and in (t_1, t_{n+k}) and so that

$$\int s\, df_h^{(k-1)} = \int s\, df_0^{(k-1)}, \qquad \text{all} \quad s \in \mathbb{S}_{k,\mathbf{t}}.$$

But then, with p_h the unique polynomial of order k which agrees with $f_h - f_0$ on $(t_i)_1^k$, the function

$$f := p_h + f_h$$

is a spline of order k with $\leqslant n$ knots in (a, b), all simple and all in (t_1, t_{n+k}), which agrees with f_0 on $\mathbf{t}$ and for which $\| df^{(k-1)} \|$ is minimal. The existence of such a spline (except for the fact that all the ξ_i's are in (t_1, t_{n+k})) could have been deduced directly from Fisher and Jerome's work [5], which considers this problem in greater generality both regarding the interpolation conditions and the seminorm minimized.

Next, consider the case in which $\mathbf{t}$ has points of multiplicity k. Proceeding as above to construct $f = p_h + f_h$, we find it still true that f is a spline function of order k with n knots $\xi_1, \ldots, \xi_n$, all simple, inside (a, b), while f agrees with f_0 on $\mathbf{t}$ and $\| df^{(k-1)} \|$ is minimal, *provided none of the* ξ_i *coincide with a point* $\mathbf{t}$ *of multiplicity* k. The contrary case cannot always be avoided as the following example shows. If $k = 2$, $a = -1$, $t_1 = t_2 = 0$, $t_3 = 1$, $b = 2$, and $f_0(t) = 1 + t - t^2$, then $\xi_1 = 0$ for all minimizing $h = \beta_1 \delta_{\xi_1}$ (note that $n = 1$) since 0 is the only point at which a nonzero $\psi \in S_{k,\mathbf{t}}$ can take on its extreme value in this case.

In view of the generality maintained in Sections 3 and 4, the preceding discussion is open to much generalization. It is possible in this fashion to analyze minimization of $\| Mf \|_p$ over $F = \{f \in \mathbb{L}_p^{(k)}[a, b] \mid \lambda_i f = \alpha_i, i = 1, \ldots, n + k\}$ with M an ordinary linear differential operator of order k in normal form with continuous coefficients on $[a, b]$ and $\lambda_1, \ldots, \lambda_{n+k}$ linear functionals which are linearly independent and continuous over $C^{(k-1)}[a, b]$, and total over ker M. Statements involving numbers of zeros, etc., would, of course, require additional hypotheses. But, this paper is already long enough.

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